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Supporting Information

for

Nonlinear features of the superconductor–ferromagnet–superconductor φ_0 Josephson junction in the ferromagnetic resonance region

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Details of calculations for Equation 22 and Equation 24

According to our approximation, at small values of the system parameters, the y component of magnetization can be determined by the general Duffing equation (Equation S1):

$$\frac{d^2 m_y}{dt^2} + 2\alpha\omega_F \frac{dm_y}{dt} + \omega_F^2 m_y - \omega_F^2 m_y^3 = \omega_F^2 G r \sin(\omega_J t). \tag{S1}$$

In Equation S1, α is a phenomenological damping constant, ω_F is the frequency of the ferromagnetic resonance, ω_J is the Josephson frequency, G is the ratio between Josephson energy and magnetic energy, and the parameter r determines the strength of the spin–orbit interaction.

In order to find frequency response function for Equation S1, we have performed the following procedure: We assume that the approximation solution of Equation S1 has the form

$$m_y = a \sin \omega_J t + b \cos \omega_J t, \tag{S2}$$

where a and b are functions of ω_J.

The first- and second-order derivatives of m_y are determined by

$$\begin{aligned} \frac{dm_y}{dt} &= a\omega_J \cos \omega_J t - b\omega_J \sin \omega_J t \\ \frac{d^2 m_y}{dt^2} &= -a\omega_J^2 \sin \omega_J t - b\omega_J^2 \cos \omega_J t. \end{aligned} \tag{S3}$$

Using trigonometric identities, the cube of m_y can be written as

$$m_y^3 = \frac{3}{4}(a^2 + b^2)[a \sin \omega_J t + b \cos \omega_J t]. \tag{S4}$$

Substituting Equation S3 and Equation S4 into Equation S1 and equating the coefficients at $\sin \omega_J t$ and $\cos \omega_J t$, we find

$$[\omega_F^2 - \omega_J^2 - \frac{3}{4}\omega_F^2 (m_y^{max})^2] a - 2\alpha\omega_F\omega_J b = \omega_F^2 Gr$$
$$, 2\alpha\omega_F\omega_J a + [\omega_F^2 - \omega_J^2 - \frac{3}{4}\omega_F^2 (m_y^{max})^2] b = 0, \tag{S5}$$

where $(m_y^{max})^2 = a^2 + b^2$. Squaring both side of the system in Equation S5 and summing them, we get the frequency response function,

$$(m_y^{max})^2 = \frac{\left(\omega_F^2 Gr\right)^2}{\left[\omega_J^2 - \omega_F^2 + \frac{3}{4}\omega_F^2 (m_y^{max})^2\right]^2 + \left(2\alpha\omega_F\omega_J\right)^2}. \tag{S6}$$

Introducing $\omega = \omega_J/\omega_F$, we can rewrite it in a simple form:

$$(m_y^{max})^2 = \frac{\left(Gr\right)^2}{\left[\omega^2 - 1 + \frac{3}{4}(m_y^{max})^2\right]^2 + \left(2\alpha\omega\right)^2}. \tag{S7}$$

In Figure S1, the frequency dependence of the amplitude corresponding to the expression in Equation S7 for values of $\alpha = 0.015, 0.02, 0.03, 0.04$ are demonstrated.

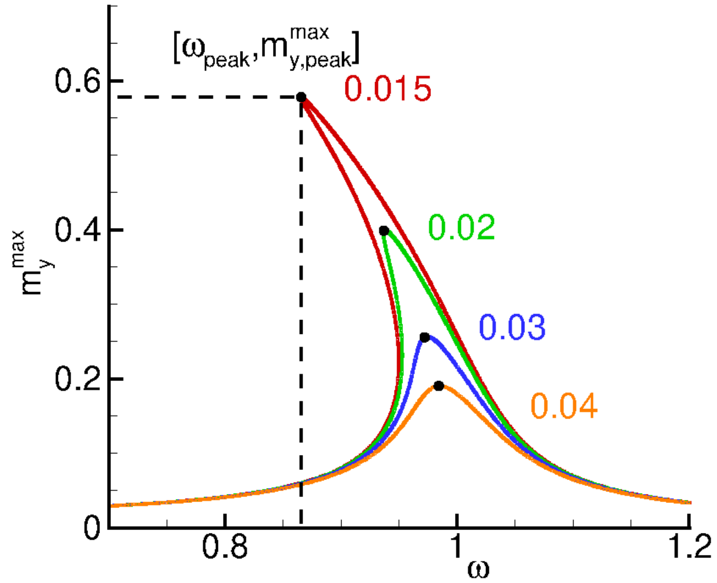

Figure S1: Frequency response in Equation S7, that is, m_y^{max} versus ω for $G = 0.05$, $r = 0.05$, and $\omega_F = 0.5$. Here, the numbers indicate the value of α.

To find the analytical anomalous damping dependence, we need to differentiate both side of Equation S7 with respect to ω and equate $d(m_y^{max})/d\omega$ to zero. We find

$$(m_{y,peak}^{max})^2 = \frac{4}{3}(1 - \omega_{peak}^2 - 2\alpha^2). \tag{S8}$$

Here $m_{y,peak}^{max}$ is the resonance peak and ω_{peak} is its position as shown in Figure S1. Finally, substituting Equation S8 into Equation S7, we obtain

$$\frac{4}{3}(1 - \omega_{peak}^2 - 2\alpha^2) = \frac{(Gr)^2}{(-2\alpha^2)^2 + (2\alpha\omega_{peak})^2}. \tag{S9}$$

After simplification of this expression, it can be written as

$$\frac{16}{3}\alpha^2(\alpha^2 + \omega_{peak}^2 - 3\alpha^2\omega_{peak}^2 - \omega_{peak}^4 - 2\alpha^4) = (Gr)^2. \tag{S10}$$

The solution of Equation S10 with respect to ω has the form

$$\omega_{peak} = \sqrt{\frac{1 - 3\alpha^2}{2} + \frac{1}{2}\sqrt{(1 - \alpha^2)^2 - 12(\frac{Gr}{4\alpha})^2}}. \tag{S11}$$

Thus, it is an analytical expression for ADD and its plot is shown in Figure S2 for $G = 0.05$ and $r = 0.05$.

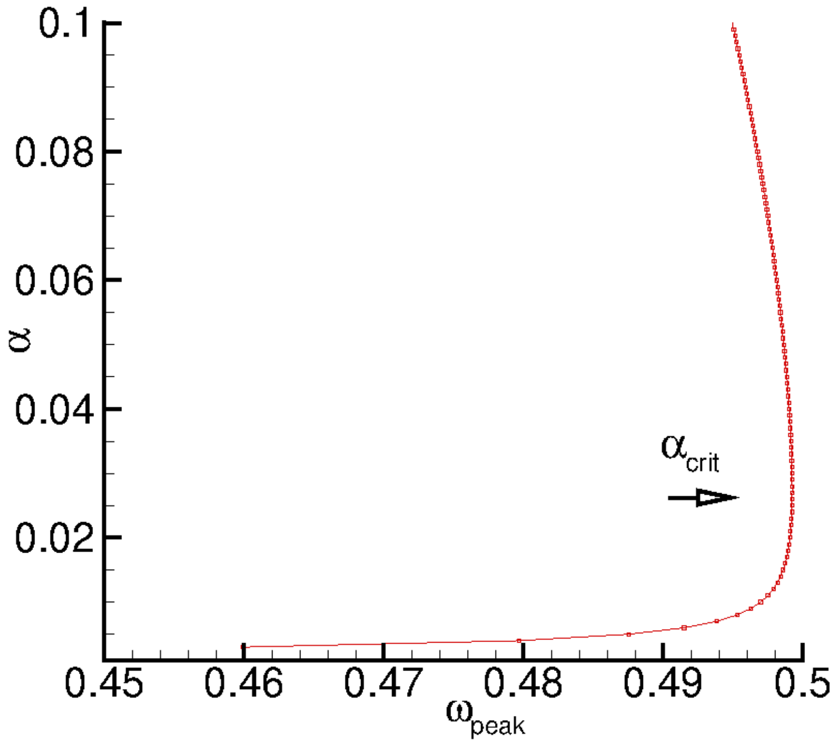

Figure S2: The resonance peak position depending on α for $G = 0.05$ and $r = 0.05$ and $\omega_F = 0.5$.

We can also find the expression for the critical value of α. To get it, we perform the following procedure: After taking the derivative of Equation S10 with respect to α, we equate $\partial\omega_{peak}/\partial\alpha$ to zero and obtain the expression for α_{crit}:

$$6\alpha_{crit}^4 - 2\alpha_{crit}^2(1 - 3\omega_{peak}^2) - \omega_{peak}^2 + \omega_{peak}^4 = 0. \tag{S12}$$

Substituting ω_{peak} in Equation S12 from Equation S11, we get

$$9\left(\frac{Gr}{4\alpha_{crit}}\right)^4 + 3\alpha_{crit}^2(10\alpha_{crit}^2 - 1)\left(\frac{Gr}{4\alpha_{crit}}\right)^2 - 2\alpha_{crit}^4(\alpha_{crit}^2 - 1)^2 = 0. \tag{S13}$$

Taking into account $10\alpha_{crit}^2 << 1$ and $\alpha_{crit}^2 << 1$, Equation S13 can be rewritten as

$$9\left(\frac{Gr}{4\alpha_{crit}}\right)^4 - 3\alpha_{crit}^2\left(\frac{Gr}{4\alpha_{crit}}\right)^2 - 2\alpha_{crit}^4 = 0. \tag{S14}$$

The solution of Equation S14 has the form

$$\left(\frac{Gr}{4\alpha_{crit}}\right)^2 = \frac{3\alpha_{crit}^2 \pm \sqrt{9\alpha_{crit}^4 + 72\alpha_{crit}^4}}{18} \tag{S15}$$

or

$$\left(\frac{Gr}{4\alpha_{crit}}\right)^2 = \frac{\alpha_{crit}^2 \pm 3\alpha_{crit}^2}{6}. \tag{S16}$$

From here we can find

$$\alpha_{crit} = \frac{1}{2}\sqrt{\sqrt{\frac{3}{2}}Gr}. \tag{S17}$$