Supporting Information

for

Effective sensor properties and sensitivity considerations of a
dynamic co-resonantly coupled cantilever sensor

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Details on mathematical derivations
Derivation of frequency shift for dynamic-mode cantilever sensors

The frequency shift \( \Delta \omega \) for a dynamic-mode cantilever sensor is generally given as:

\[
\Delta \omega = \sqrt{\frac{k + \Delta k}{m_{\text{eff}} + \Delta m}} - \sqrt{\frac{k}{m_{\text{eff}}}},
\]

where \( k \) and \( m_{\text{eff}} \) are the cantilever’s spring constant and effective mass, respectively and \( \Delta k \) and \( \Delta m \) denote small changes due to external interactions (force gradient or mass addition).

For negligible mass change, i.e. \( \Delta m = 0 \), the frequency shift in dependence of a force gradient \( \Delta k \) can be derived by applying the approximation \([1]\):

\[
\sqrt{1 + \epsilon} \approx \frac{\epsilon}{2} + 1, \quad \epsilon \ll 1
\]

for small \( \Delta k \) to the first term in equation (S.1). This results in:

\[
\Delta \omega = \sqrt{\frac{k (1 + \frac{\Delta k}{k})}{m_{\text{eff}}}} - \sqrt{\frac{k}{m_{\text{eff}}}}
\]

(S.3)

\[
\Delta \omega = \sqrt{\frac{k}{m_{\text{eff}}} \cdot \sqrt{1 + \frac{\Delta k}{k}} - \sqrt{\frac{k}{m_{\text{eff}}}}}
\]

(S.4)

\[
\Delta \omega \approx \sqrt{\frac{k}{m_{\text{eff}}} \cdot \left( \frac{\Delta k}{2k} + 1 \right) - \sqrt{\frac{k}{m_{\text{eff}}}}}
\]

(S.5)

\[
\Delta \omega \approx \sqrt{\frac{k}{m_{\text{eff}}} \cdot \frac{\Delta k}{2k} + \frac{k}{m_{\text{eff}}} - \sqrt{\frac{k}{m_{\text{eff}}}}}
\]

(S.6)

\[
\Delta \omega \approx \omega_0 \cdot \frac{\Delta k}{2k}
\]

(S.7)

For negligible \( \Delta k \), a similar expression can be found for the frequency shift in dependence on the mass change \( \Delta m \). In this case a Taylor series expansion is employed under the assumption that \( \Delta m \) is small, i.e. at the point \( \Delta m = 0 \). To do so, equation (S.1) is rewritten:

\[
\Delta \omega (\Delta m, 0) = \sqrt{k \cdot (m_{\text{eff}} + \Delta m)^{-1/2}} - \sqrt{\frac{k}{m_{\text{eff}}}}.
\]

(S.8)

It is sufficient to consider only the first two terms (stationary and first derivative) of the expansion:
\[ \Delta \omega (\Delta m, 0) \approx \Delta \omega (0) + \frac{\partial \Delta \omega (0)}{\partial \Delta m} \cdot (\Delta m - 0) \quad (S.9) \]
\[ \Delta \omega (\Delta m, 0) \approx 0 - \frac{1}{2} \sqrt{k (m_{eff})^{-3/2}} \cdot \Delta m \quad (S.10) \]
\[ \Delta \omega (\Delta m, 0) \approx -\frac{\Delta m}{2 m_{eff}} \cdot \sqrt{\frac{k}{m_{eff}}} \quad (S.11) \]
\[ \Delta \omega (\Delta m, 0) \approx -\Delta m \cdot \omega_0 \quad (S.12) \]

**Effective properties of the coupled system in dependence on eigenfrequency deviation**

**Resonance frequencies**

The resonance frequencies of the coupled system can be expressed as a function of the eigenfrequency deviation \( \Delta \omega_{eigen} = (\omega_2 - \omega_1)/\omega_1 \):

\[ \omega_{a,b}^2 = \omega_1^2 \cdot \left[ \frac{1}{2} \left( K_{12} + K_{23} \cdot \Omega^2 \right) \mp \sqrt{\frac{1}{4} \left( K_{12} - K_{23} \cdot \Omega^2 \right)^2 + \frac{k_2}{k_1} \Omega^2} \right] \quad , \quad (S.13) \]

with

\[ \Omega = 1 + \Delta \omega_{eigen} \quad . \quad (S.14) \]

In that case, the eigenfrequency of the bigger oscillator (1) is assumed to be fixed and only that of the nanocantilever is varied. Please note that this is just a different form of equation (11) from the main text.

**Effective spring constants**

Based on equation (S.13) and equations (15) and (16) from the main paper, the effective spring constants can also be expressed in dependence on the eigenfrequency deviation, leading to:

\[ k_{a,b}^{eff} = k_3 \frac{2}{3} \sqrt{\frac{1}{2} \left( K_{12} + \Omega^2 \right) \mp \sqrt{\frac{1}{4} \left( K_{12} - \Omega^2 \right)^2 + \frac{k_2}{k_1} \Omega^2}} \quad , \quad (S.15) \]

with

\[ A = \sqrt{\frac{1}{2} \left( K_{12} + K_{23} \cdot \Omega^2 \right) \mp \sqrt{\frac{1}{4} \left( K_{12} - K_{23} \cdot \Omega^2 \right)^2 + \frac{k_2}{k_1} \Omega^2}} - \sqrt{\frac{1}{2} \left( K_{12} + \Omega^2 \right) \mp \sqrt{\frac{1}{4} \left( K_{12} - \Omega^2 \right)^2 + \frac{k_2}{k_1} \Omega^2}} \quad . \quad (S.16) \]
References